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Operator basis for analytic signal construction

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Abstract The operator basis for one-dimensional analytical signal construction is considered. Why the Hilbert transform plays a fundamental role is clarified. Extension to the case of multidimensional signals is considered. Possible ramifications for the definition of a fractional analytic signal are discussed.

Keywords Hilbert transform · Analytic signal · Multidimensional analytic signal · Fractional analytic signal

1 Introduction

The analytic signal arising from the function g in the time domain is defined in the following way:

$$f(t) = g(t) + iHg(t), \tag{1}$$

where H denotes the Hilbert transform operator, defined on the real line $\mathbb{R}$, by

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)\mathrm{d}y}{x-y}, \text{ for } x \in \mathbb{R}, \tag{2}$$

and P signifies that the Cauchy principal value is taken. The construction of a complex signal can be considered in a more abstract setting. A principal problem is to determine an operator $\mathcal{O}$ such that a signal written in the form

$$w(t) = u(t) + iv(t), \tag{3}$$

is constructed using

$$v(t) = \mathcal{O}u(t). \tag{4}$$

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What requirements must be placed on $\mathcal{O}$ such that the resulting complex signal has certain desirable properties? Vakman (1996, 1997) and Vakman and Vaĭnshteĭn (1977) proposed three conditions that should be satisfied by the operator $\mathcal{O}$. It should incorporate an appropriate response to a small perturabation of the signal; it should handle constraints on the phase and frequency when the signal is multiplied by a scalar, and third, that the amplitude and frequency of a sinusoidal signal should be preserved under the action of the operator $\mathcal{O}$. The Hilbert transform satisfies these three conditions. Vakman's approach leads to other issues for consideration. What might constitute the most useful definition of the instantaneous amplitude and the instantaneous frequency? What boundedness relationships exist between the signal and amplitude (Loughlin 1998)?

The focus of this work is to examine if conditions other than those proposed by Vakman, might serve as a useful basis for defining an analytic signal operator. Section 2 considers the one-dimensional case. The generalization to n-dimensional signals is treated in Sect. 3. Section 4 discusses the situation for a fractional analytical signal operator.

2 Translation invariance and homogeneity conditions

A fundamental question is to ascertain to what extent invariance conditions might determine the form of $\mathcal{O}$ in Eq. 4. In this section, both a translation invariance restriction and a homogeneity condition are examined. Signals satisfying a translational invariance condition are much easier to analyze. There are important implications (via the Fourier convolution theorem) for the treatment of such signals in the frequency domain. The translational invariance condition, together with causal arguments, provides a pathway to a formal definition of an analytic signal operator. The homogeneity condition finds application for signals for which the signal and a rescaled copy of the signal, with appropriate signal amplitude renormalization, are identical. In the sequel, it will be demonstrated that these two invariance conditions are sufficient to fix the form of the analytic signal operator.

Under the action of a translation of the signal $u(t)$, it is reasonable to expect that the signal satisfies:

$$u(t) \to u(t-a), \quad \text{then } w(t) \to w(t-a) = u(t-a) + iv(t-a). \tag{5}$$

Since $v(t)$ is determined from $u(t)$, it is required that the following sequence apply:

$$\mathcal{O}u(t-a) = \mathcal{O}\tau_a u(t), \tag{6}$$

$$\mathcal{O}\tau_a u(t) = \tau_a \mathcal{O}u(t), \tag{7}$$

$$\tau_a \mathcal{O}u(t) = v(t-a), \tag{8}$$

where τ_a is the translation operator defined by

$$\tau_a f(x) = f(x-a), \quad \text{for } a \in \mathbb{R}. \tag{9}$$

An operator T which satisfies

$$T\tau_a f(x) = \tau_a T f(x) \tag{10}$$

is termed a *translation-invariant* operator.

In a related fashion, for a constant $b > 0$, it is reasonable to expect that the signal satisfies:

$$u(t) \to u(bt), \quad \text{then } w(t) \to w(bt) = u(bt) + iv(bt). \tag{11}$$

It is desirable that the following sequence applies:

$$\mathcal{O}u(bt) = \mathcal{O}S_b u(t), \tag{12}$$

$$\mathcal{O}S_b u(t) = S_b \mathcal{O}u(t), \tag{13}$$

$$S_b \mathcal{O}u(t) = v(bt), \tag{14}$$

where S_b is the dilation operator (homothetic operator) defined by

$$S_b f(x) = f(bx), \quad \text{for } b > 0. \tag{15}$$

The two key requirements imposed in Eqs. 7 and 13 are the commutator conditions:

$$[\mathcal{O}, \tau_a] \equiv \mathcal{O}\tau_a - \tau_a \mathcal{O} = 0, \tag{16}$$

and

$$[\mathcal{O}, S_b] \equiv \mathcal{O}S_b - S_b \mathcal{O} = 0. \tag{17}$$

If $\mathcal{O}$ is an integral operator, then its kernel function $k(x, y)$ must be of the form of a difference kernel, $k(x, y) = k(x - y, 0) = k(0, y - x)$, if Eq. 16 is to be satisfied. For Eq. 17 to hold, the kernel must satisfy a homogeneity condition, that is $k(mx, my) = m^{-1}k(x, y)$, for $m > 0$.

An important result, apparently not well-known outside the Mathematics literature, is the following theorem of McLean and Elliott (1988) and Stein (1970): If $\mathcal{O}$ is a bounded linear operator acting on functions of the class $L^p(\mathbb{R})$ for $1 < p < \infty$, and if $\mathcal{O}$ is required to commute with both the translation operator and the dilation operator, for positive dilations, then $\mathcal{O}$ takes the form:

$$\mathcal{O} = \alpha I + \beta H, \tag{18}$$

where α and β are constants, and I is the unit operator. With the proper selection of the constants in Eq. 18, then the very general constraints of having a translation-invariant operator and the commutator condition $[\mathcal{O}, S_b] = 0$ for $b > 0$, leads to the identification of $\mathcal{O}$ in the equation $v(t) = \mathcal{O}u(t)$ with the Hilbert transform operator. With the appropriate choice for α, this is the only operator that satisfies the invariance conditions and the Valkman conditions. The choice $\alpha = 1$ and $\beta = i$ in Eq. 18 gives rise to the definition of the analytic signal operator A, as

$$A = I + iH, \tag{19}$$

so that if $z(t)$ denotes the analytic signal corresponding to the real signal $s(t)$, then

$$z(t) \equiv (As)(t). \tag{20}$$

3 *n*-Dimensional signals

To treat n-dimensional signals, we require the n-dimensional Hilbert transform, which is defined by Pandey (1996)

$$H_n f(x_1, x_2, x_3, \ldots, x_n) = \frac{1}{\pi^n} P \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f(s_1, s_2, s_3, \ldots, s_n)\, \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3 \ldots \mathrm{d}s_n}{\prod_{k=1}^{n}(x_k - s_k)}. \tag{21}$$

This definition is a specialization of a more general class of singular-integral operators called Calderón-Zygmund operators: see Meyer and Coifman (1997). The n-dimensional Hilbert transform operator can be expressed in terms of a product of one-dimensional Hilbert transform operators. The variable on which the one-dimensional operator acts is specified by a subscript, thus $H_{(k)}$. So H_n can be written as

$$H_n = \prod_{k=1}^{n} H_{(k)}, \tag{22}$$

where

$$\begin{aligned} &H_{(k)} f(s_1, s_2, \ldots, s_{k-1}, x_k, s_{k+1}, \ldots, s_n) \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s_1, s_2, \ldots, s_{k-1}, s_k, s_{k+1}, \ldots, s_n)\, \mathrm{d}s_k}{x_k - s_k}. \end{aligned} \tag{23}$$

The theorem of McLean and Elliot has an extension to the n-dimensional case (see ref. Pandey (1996), p. 188): If $\mathcal{O}$ is a bounded linear operator acting on functions of the class $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and if $\mathcal{O}$ commutes with the translation and dilation operators, then by an extension of the arguments of Sect. 2, we have

$$\mathcal{O} = \alpha I + \sum_{k=1}^{n} \beta_k H_{(k)} + \sum_{j=1}^{n} \sum_{k>j}^{n} \gamma_{jk} H_{(j)} H_{(k)} + \cdots + \omega H_n, \tag{24}$$

where $\alpha, \beta_k, \gamma_{jk}$, and ω are constants, I is the identity operator in $\mathbb{R}^n$ and H_n denotes the n-dimensional Hilbert transform operator defined in Eq. 21. An n-dimensional analytic operator A_n can be defined by reference to Eq. 24.

4 The fractional analytical signal in one dimension

The notion of fractional Hilbert transform has emerged over the last several years as a means for enhanced image reconstruction. Zayed introduced a generalization of the Hilbert transform which might rightly be called a fractional Hilbert transform (Zayed 1998). His definition is (Zayed 1998)

$$H_\alpha f(t) = \frac{e^{-\frac{i}{2}\cot\alpha\, t^2}}{\pi} P \int_{-\infty}^{\infty} \frac{f(x) e^{\frac{i}{2}\cot\alpha\, x^2}\, \mathrm{d}x}{t - x}, \tag{25}$$

for $\alpha \neq 0, \frac{\pi}{2}$, or π. For the case $\alpha = (2n+1)\pi/2$ with $n \in \mathbb{Z}$, Eq. 25 reduces back to the standard definition of the Hilbert transform. A key result that is obtained from this definition is the following:

$$\mathcal{F}_\alpha H_\alpha s(u) = -i \operatorname{sgn} u\, \mathcal{F}_\alpha s(u), \quad \text{for } 0 < \alpha < \pi. \tag{26}$$

For $\pi < \alpha < 2\pi$ the same result holds with the sign of the right hand side of the equation reversed. Equation 26 has the same form as the important result:

$$\mathcal{F}\, Hf(x) = -i \operatorname{sgn} x \mathcal{F} f(x). \tag{27}$$

An alternative definition of the fractional Hilbert transform has been given by Cusmariu (2002). He defined the fractional Hilbert transform operator directly in terms of the normal Hilbert transform operator by the relation:

$$H_\alpha = \cos\alpha + \sin\alpha H. \tag{28}$$

Lohmann and coworkers (1996a, b) have also introduced a definition for the fractional Hilbert transform, and discussed its implementation. Their definition has a structure related to that in Eq. 28. A discrete version of the fractional Hilbert transform has been considered by Pei and Yeh (1998, 2000). An advantage of the definition in Eq. 28 is that the mathematical forms of several of the standard properties of the Hilbert transform are preserved. The Cusmariu definition of the fractional Hilbert transform satisfies

$$H_\alpha H_\beta = H_{\alpha+\beta}, \tag{29}$$

and hence a semigroup condition under composition holds for the operator H_α. For a signal $s(t)$, the following holds:

$$\int_{-\infty}^{\infty} s^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} [H_\alpha s(t)]^2\,\mathrm{d}t. \tag{30}$$

This result indicates that the action of the fractional Hilbert transform on the signal preserves the energy of the signal. The analytic signal $z(t)$ corresponding to $s(t)$ is an eigenfunction of H_α:

$$H_\alpha z(t) = e^{-i\alpha} z(t). \tag{31}$$

At least four different suggestions for the definition of the fractional analytic signal have been proposed. Zayed has defined the fractional analytical signal as (Zayed 1998)

$$z_{1\alpha}(t) = s(t) + iH_\alpha s(t), \tag{32}$$

where H_α is defined in Eq. 25. If $0 < \alpha < \pi$, then it follows from this definition that $Z_{1\alpha}(\omega)$ vanishes for $\omega < 0$. To see this, take the fractional Fourier transform of order α of Eq. 32 and employ Eq. 26, then

$$Z_{1\alpha}(\omega) = S_\alpha(\omega)\{1 + \operatorname{sgn}\omega\}, \tag{33}$$

which is the analog of the result for the non-fractional case.

Cusmariu has suggested three different possible definitions based on Eq. 28 (Cusmariu 2002). The first is similar to Zayed's definition, but with Eq. 28 employed in place of Eq. 25:

$$z_{2\alpha}(t) = s(t) + iH_\alpha s(t). \tag{34}$$

Cusmariu's second suggestion was to define the fractional analytic signal by the result:

$$z_{3\alpha}(t) = H_\alpha z(t). \tag{35}$$

This choice has some geometric appeal. It can be easily shown that $|z_{3\alpha}(t)|^2 = |z(t)|^2$, and by taking the Fourier transform of Eq. 35, that $|Z_{3\alpha}(\omega)| = |Z(\omega)|$. The third suggestion of Cusmariu involves the definition

$$z_{4\alpha}(t) = \cos\alpha s(t) + i\sin\alpha H s(t). \tag{36}$$

Which of these definitions of the fractional analytic signal will turn out to be the most useful in practical applications is yet to be determined. However, in the context of preserving translational invariance and the commutative condition for the dilation operator, we can make a distinction between the available choices.

Using the definition in Eq. 28, then clearly

$$[H_\alpha, \tau_a] = 0, \tag{37}$$

and

$$[H_\alpha, S_b] = 0. \tag{38}$$

Therefore, the three Cusmariu definitions of the fractional analytic signal operator all satisfy the translational invariance condition of Eq. 7 and the commutator condition of Eq. 13.

Making use of Eq. 25, then

$$\begin{aligned}\{H_\alpha \tau_a f(x)\}(t) &= \frac{e^{-\frac{i}{2}\cot\alpha\, t^2}}{\pi} P \int_{-\infty}^{\infty} \frac{f(x-a)e^{\frac{i}{2}\cot\alpha\, x^2}\, \mathrm{d}x}{t-x} \\ &= \frac{e^{-\frac{i}{2}\cot\alpha\, t^2}}{\pi} P \int_{-\infty}^{\infty} \frac{f(x)e^{\frac{i}{2}\cot\alpha\, (x+a)^2}\, \mathrm{d}x}{t-a-x} \\ &\neq \tau_a\{H_\alpha f(x)\}(t), \end{aligned} \tag{39}$$

so the Zayed definition given in Eq. 25 does not satisfy Eq. 37. The multiplication of the original signal by a chirp function breaks the translational invariance. A further calculation shows that Eq. 38 does not hold when H_α is defined by Eq. 25. The fractional analytic signal defined in Eq. 32 therefore represents a departure from Eqs. 7 to 13.

Forcing the fractional Hilbert transform to satisfy the invariance properties given in Eqs. 37 and 38, and defining a fractional analytic signal directly in terms of H_α, will at least guarantee that there is a preservation of the invariance properties for A_α, rather than producing an abrupt change, as α moves away from the value zero. Lost of translation invariance would lead to additional complications in dealing with the signal: for an example of some of the issues, see Hogan and Lakey (2005). Future practical applications might present compelling reasons not to force these invariance properties. Probably the most important area where the invariance ideas might play a role is in image processing, in particular, in image edge enhancement and in image compression problems.

5 Conclusion

For one-dimensional signals, the requirements of translational invariance and signal homogeneity are sufficient to fix to within a pair of arbitrary constants, the definition of the analytic signal operator. For an n-dimensional signal, translational invariance and signal homogeneity also lead to an operator form for the analytic signal operator, with some arbitrary constants appearing. In both cases, the Hilbert transform plays the key role.

Whether or not it is useful to retain the conditions given in Eqs. 7 and 13, when formulating a definition of a fractional analytic signal operator, is yet to be determined by the practical applications that can be made. Employing these conditions in the formulation of a fractional Hilbert transform operator, does have the advantage that the fractional analytic signal goes smoothly to the normal analytic signal, and stays within a translation-invariant space.

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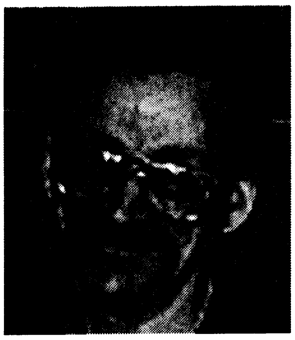
Author Biography



Frederick W. King received his undergraduate degree from the University of Sydney in 1969, his M. Sc. from the University of Calgary in 1971, and a PhD from Queens University in Canada in 1975. He did postdoctoral work at Oxford University, Northwestern University, and Brock University. He joined the University of Wisconsin-Eau Claire in 1979, where he is currently a Professor in the Chemistry Department. He has received two awards from the Dreyfus Foundation. His research interests have dealt with theoretical problems in a variety of different areas.