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Sum rules for the forward elastic low energy scattering of light

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Abstract. Sum rule constraints have been obtained for the scattering amplitude, differential elastic scattering cross section and the total cross section for light scattering at low energies. An application of one of the sum rules is made for the case of photon scattering from the H^- ion, calculated in the asymptotic approximation.

1. Introduction

A number of interesting sum rule relations for the total cross section and the differential elastic cross section have been derived from the dispersion relations for the forward scattering of light. This has been a particularly stimulating area for high energy photon scattering (Damashek and Gilman 1970, Drell and Hearn 1966, Ferro Fontán *et al* 1972, Maximon and O'Connell 1974). On the other hand, low energy scattering has received considerably less attention from the time Gell-Mann *et al* (1954) succeeded in providing a dispersion theoretic derivation of the Thomas–Reiche–Kuhn sum rule (see also Goldberger and Low 1968).

The motivation for obtaining sum rules applicable to low energy scattering is provided by recent theoretical interest in photon scattering from small atomic systems $H^{\cdot}$ and H^- (Gavrila 1967, Granovskii 1969, Gavrila and Costescu 1970, Moses 1972, 1973, Adelman 1973). Very few consistency checks are available to test approximate theoretical relationships for photon scattering. Additionally, sum rule constraints provide a criterion for determining the quality of experimentally observed cross sections. The H^- ion provides a suitable candidate for the testing of sum rules of the type discussed in this paper. Within the asymptotic approximation, fairly simple expressions for the differential elastic cross section and the total cross section of the H^- ion can be derived, and the consistency of these results can be clarified by the application of sum rule constraints.

The sum rules are based on the usual assumptions which allow the dispersion relations to be derived. It is of course implicitly understood that writing the upper limit of the definite integrals as infinity is only formal since, within the present context, the sum rules are intended only for scattering from bound electrons. This is not the case at sufficiently high energies where Compton and Delbrück scattering and pair production become increasingly important. The problem of photon scattering from bound electrons in the very high energy region has been considered by Goldberger and Low (1968).

2. Derivation of sum rules

The starting point for work on sum rule constraints are the well known Kramers–

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Kronig relationships, connecting the real and imaginary parts of the forward scattering amplitude, denoted by $f_{\mathrm{R}}(k)$ and $f_{\mathrm{I}}(k)$ respectively. These relations take the form

$$f_{\mathrm{R}}(k) = \frac{2P}{\pi}\int_0^\infty \frac{k' f_{\mathrm{I}}(k')\,\mathrm{d}k'}{k'^2-k^2}, \tag{2.1}$$

$$f_{\mathrm{I}}(k) = \frac{-2k}{\pi}P\int_0^\infty \frac{f_{\mathrm{R}}(k')\,\mathrm{d}k'}{k'^2-k^2}. \tag{2.2}$$

An approach which has been of considerable utility in other applications, namely obtaining suitable constraints on the optical constants, has been the examination and comparison of the asymptotic behaviour of the appropriate dispersion relations (Altarelli *et al* 1972, Altarelli and Smith 1974). A number of useful sum rules have recently emerged from this approach. This idea is taken up for the scattering amplitude for the forward elastic photon scattering from bound electron systems.

The asymptotic behaviour of the scattering amplitude takes the form

$$f(k) \simeq A+Bk^{-2} \qquad \text{as } k\to\infty \tag{2.3}$$

where A and B are constants. This asymptotic behaviour is obtained from the Kramers–Heisenberg formula for $f(k)$ in the dipole approximation. This choice requires us to restrict k to satisfy the condition $k \ll r_{\mathrm{a}}^{-1}$, where r_{a} is the molecular dimension of the atom or molecule from which scattering is being considered. In this section dispersion relations based on the functions $k^{-1}f(k), k^{-2}f(k)$ and their squares will be considered. These functions are well behaved in the limit $k\to 0$. Along with the asymmptotic behaviour of the scattering amplitude, we also require the asymptotic behaviour of the principal value integral

$$g(y) = P\int_0^\infty \frac{F(x)\,\mathrm{d}x}{y-x} \qquad \text{for } y\to\infty \tag{2.4}$$

where $F(x)$ is a continuous differentiable function. The asymptotic behaviour of $g(y)$ has been studied by Frye and Warnock (1963). The following results have been obtained (Frye and Warnock 1963):

$$g(y) = \frac{1}{y}\int_0^\infty F(x)\,\mathrm{d}x + \mathrm{O}(y^{-1}\ln^{1-\alpha}y) \tag{2.5}$$

for

$$F(x) = \mathrm{O}(x^{-1}\ln^{-\alpha}x), \qquad \alpha>1; \tag{2.6}$$

$$g(y) = \frac{1}{y}\int_0^\infty F(x)\,\mathrm{d}x + \mathrm{O}(y^{-\alpha}) \tag{2.7}$$

for

$$F(x) = \mathrm{O}(x^{-\alpha}), \qquad \alpha>1\ (\alpha\neq 2); \tag{2.8}$$

and (Altarelli *et al* 1972):

$$g(y) = \frac{1}{y}\int_0^\infty F(x)\,\mathrm{d}x + \mathrm{O}(y^{-2}\ln y) \tag{2.9}$$

for

$$F(x) = \mathrm{O}(x^{-2}). \tag{2.10}$$

As a reminder to the reader, the notation $F(x) = O(h(x))$ as $x \to \infty$ means that there exist positive constants a, b such that $|F(x)| < a|h(x)|$ for $x \geqslant b$. Kubo and Ichimura (1972) have considered the expansion of the principal value integral in a more general way. By inserting the expansion

$$\frac{1}{y-x} = \sum_{n=0}^{\infty} \frac{x^n}{y^{n+1}} \tag{2.11}$$

into (2.4), then

$$g(y) = \sum_{n=0}^{\infty} \frac{1}{y^{n+1}} \int_0^{\infty} x^n F(x)\,\mathrm{d}x \tag{2.12}$$

which is to be regarded as a purely formal expansion and as pointed out by Kubo and Ichimura, may not make any sense and may not be convergent.

For the elastic scattering of light from bound electronic systems, $f(k)$ does not vanish as $k \to \infty$ (see equation (2.3)), so the Kramers–Kronig relations (2.1) and (2.2) require modification. Dispersion relations based on the function $k^{-1}f(k)$ lead to the results

$$f_{\mathrm{R}}(k) = \frac{2k^2}{\pi} P \int_0^{\infty} \frac{f_{\mathrm{I}}(k')\,\mathrm{d}k'}{k'(k'^2-k^2)} \tag{2.13}$$

and

$$f_{\mathrm{I}}(k) = \frac{-2kP}{\pi} \int_0^{\infty} \frac{f_{\mathrm{R}}(k')\,\mathrm{d}k'}{(k'^2-k^2)}. \tag{2.14}$$

Dispersion relations based on the function $k^{-2}f(k)$ will also be employed; they are

$$f_{\mathrm{R}}(k) = \frac{2k^2 P}{\pi} \int_0^{\infty} \frac{f_{\mathrm{I}}(k')\,\mathrm{d}k'}{k'(k'^2-k^2)} \tag{2.15}$$

and

$$f_{\mathrm{I}}(k) = \frac{-2k^3}{\pi} P \int_0^{\infty} \frac{f_{\mathrm{R}}(k')\,\mathrm{d}k'}{k'^2(k'^2-k^2)}. \tag{2.16}$$

Note that there is no need to indent the path of integration at $k = 0$, since the function $k^{-2}f(k)$ is well behaved at $k = 0$. Finally, we will make use of the dispersion relations based on $k^{-2}f^2(k)$, which are

$$f_{\mathrm{R}}^2(k) - f_{\mathrm{I}}^2(k) = \frac{4k^2}{\pi} P \int_0^{\infty} \frac{f_{\mathrm{I}}(k') f_{\mathrm{R}}(k')\,\mathrm{d}k'}{k'(k'^2-k^2)} \tag{2.17}$$

and

$$f_{\mathrm{I}}(k) f_{\mathrm{R}}(k) = \frac{-k^3}{\pi} P \int_0^{\infty} \frac{[f_{\mathrm{R}}^2(k') - f_{\mathrm{I}}^2(k')]\,\mathrm{d}k'.}{k'^2(k'^2-k^2)} \tag{2.18}$$

We shall consider first the asymptotic limit of equation (2.18). If (2.18) is rearranged to the format of (2.4), then the leading term of $F(x)$ is of the form $x^{-3/2}$, which allows the application of (2.7). Hence (2.18) becomes

$$\pi k^{-3} f_{\mathrm{I}}(k) f_{\mathrm{R}}(k) = k^{-2} \int_0^{\infty} \frac{[f_{\mathrm{R}}^2(k') - f_{\mathrm{I}}^2(k')]\,\mathrm{d}k'}{k'^2} + O(k^{-3}) \qquad \text{for } k \to \infty. \tag{2.19}$$

From the asymptotic form of $f(k)$, equation (2.3), we have the result

$$\pi k^{-3} f_{\mathrm{I}}(k) f_{\mathrm{R}}(k) \simeq \mathrm{O}(k^{-3}), \qquad k \to \infty. \tag{2.20}$$

Comparison of the asymptotic expansion of both sides of equation (2.19) leads to the expression

$$\int_0^\infty [f_{\mathrm{R}}^2(k) - f_{\mathrm{I}}^2(k)] k^{-2} \, \mathrm{d}k = 0. \tag{2.21}$$

This result can be recast into a more useful form by writing

$$\int_0^\infty [f_{\mathrm{R}}^2(k) + f_{\mathrm{I}}^2(k)] k^{-2} \, \mathrm{d}k = 2 \int_0^\infty f_{\mathrm{I}}^2(k) k^{-2} \, \mathrm{d}k \tag{2.22}$$

and employing the expressions for the differential elastic scattering cross section

$$\frac{\mathrm{d}\sigma_{\mathrm{e}}(k)}{\mathrm{d}\Omega} = f_{\mathrm{R}}^2(k) + f_{\mathrm{I}}^2(k) \tag{2.23}$$

and the total cross section

$$\sigma_{\mathrm{T}}(k) = \frac{4\pi f_{\mathrm{I}}(k)}{k}. \tag{2.24}$$

Equation (2.22) becomes

$$\int_0^\infty \frac{\mathrm{d}\sigma_{\mathrm{e}}(k)}{\mathrm{d}\Omega} \frac{\mathrm{d}k}{k^2} = \frac{1}{8\pi^2} \int_0^\infty \sigma_{\mathrm{T}}^2(k) \, \mathrm{d}k \tag{2.25}$$

which is a principal result of this paper. The result obtained is the low energy analogue of a result recently derived using different arguments to those above, for photon scattering from unbound electrons in the high energy domain (Maximon and O'Connell 1974).

Considering the dispersion relation based on the function $k^{-4} f^2(k)$ leads to

$$f_{\mathrm{I}}(k) f_{\mathrm{R}}(k) = \frac{-k^5 P}{\pi} \int_0^\infty \frac{[f_{\mathrm{R}}^2(k') - f_{\mathrm{I}}^2(k')] \, \mathrm{d}k'}{k'^4 (k'^2 - k^2)}. \tag{2.26}$$

In a similar manner, the above asymptotic argument applied to equation (2.26) gives the result

$$\pi k^{-5} f_{\mathrm{I}}(k) f_{\mathrm{R}}(k) = k^{-2} \int_0^\infty [f_{\mathrm{R}}^2(k') - f_{\mathrm{I}}^2(k')] k'^{-4} \, \mathrm{d}k' + \mathrm{O}(k^{-5}), \qquad k \to \infty. \tag{2.27}$$

Comparison of the asymptotic expansions as $k \to \infty$ leads to the result that

$$\int_0^\infty [f_{\mathrm{R}}^2(k) - f_{\mathrm{I}}^2(k)] k^{-4} \, \mathrm{d}k = 0 \tag{2.28}$$

which can be rearranged to read

$$\int_0^\infty \frac{\mathrm{d}\sigma_{\mathrm{e}}(k)}{\mathrm{d}\Omega} \frac{\mathrm{d}k}{k^4} = \frac{1}{8\pi^2} \int_0^\infty \sigma_{\mathrm{T}}^2(k) \frac{\mathrm{d}k}{k^2}. \tag{2.29}$$

The expression (2.29) may also be derived independently of the asymptotic expansion argument. Comparison of the dispersion relations (2.26) and (2.18) yields the result (2.29) directly.

In a completely analogous manner, sum rule relations based on higher powers of $f(k)/k$ can be derived. For example, the dispersion relation based on $k^{-3}f^3(k)$ is

$$f_R^3(k) - 3f_R(k)f_I^2(k) = \frac{2k^4}{\pi} P \int_0^\infty \frac{[3f_R^2(k')f_I(k') - f_I^3(k')]\,dk'}{k'^3(k'^2 - k^2)}. \tag{2.30}$$

Comparing the asymptotic expansion as $k \to \infty$ of both sides of (2.30) leads to the result

$$\int_0^\infty [3f_R^2(k)f_I(k) - f_I^3(k)]k^{-3}\,dk = 0 \tag{2.31}$$

which can be arranged to give

$$\int_0^\infty \sigma_T(k)\frac{d\sigma_e(k)}{d\Omega}\frac{dk}{k^2} = \frac{1}{12\pi^2}\int_0^\infty \sigma_T^3(k)\,dk. \tag{2.32}$$

A more general form of equation (2.29) can be derived directly from (2.13). From equation (2.13) we have that

$$f_R(k)f_I(k) = \frac{2k^2}{\pi} f_I(k) P \int_0^\infty \frac{f_I(k')\,dk'}{k'(k'^2 - k^2)} \tag{2.33}$$

and comparing this with (2.18) leads to the result

$$2f_I(k)P\int_0^\infty \frac{f_I(k')\,dk'}{k'(k'^2 - k^2)} + kP\int_0^\infty \frac{[f_R^2(k') - f_I^2(k')]\,dk'}{k'^2(k'^2 - k^2)} = 0 \tag{2.34}$$

and hence the expression

$$\int_0^\infty \frac{d\sigma_e(k)}{d\Omega}\frac{dk}{k^2(k^2 - k'^2)} = \frac{1}{8\pi^2}\int_0^\infty \frac{\sigma_T(k)[\sigma_T(k) - \sigma_T(k')]\,dk}{(k^2 - k'^2)}. \tag{2.35}$$

The limit $k' \to 0$ in (2.35) now yields (2.29).

An interesting procedure for obtaining *superconvergent* dispersion relations has been indicated by Liu and Okubo (1967). This approach has been applied to obtain sum rule constraints on the optical properties with success (Villani and Zimerman 1973). The essence of this approach amounts to multiplying the amplitude by a function which retains essentially the same domain for which the amplitude is a holomorphic function. This can be implemented for the scattering amplitude for photon scattering, by considering the integral

$$\oint \frac{f(z)\,e^{i\pi\beta}\,dz}{z^2(z-\alpha)^\beta(z+\alpha)^\beta} \qquad (-\tfrac{1}{2} < \beta < 1),$$

with the contour a large semicircle in the upper-half complex frequency plane, together with a strip along the real frequency axis $(-\alpha, \alpha)$ plus the portions $k + i\epsilon$ along the cuts from $-\alpha$ to $-\infty$ and α to ∞. Evaluation of the integral leads to the result

$$\sin\pi\beta\int_\alpha^\infty \frac{\sigma_T(k)\,dk}{k(k-\alpha)^\beta(k+\alpha)^\beta}$$
$$= 4\pi\cos\pi\beta\int_\alpha^\infty \frac{f_R(k)\,dk}{k^2(k-\alpha)^\beta(k+\alpha)^\beta} + 4\pi\int_0^\alpha \frac{f_R(k)\,dk}{k^2(\alpha+k)^\beta(\alpha-k)^\beta} \tag{2.36}$$

which simplifies for the special cases $\beta = \frac{1}{2}$ and $\beta = 0$ to

$$\int_\alpha^\infty \frac{\sigma_T(k)\,dk}{k(k^2-\alpha^2)^{1/2}} = 4\pi \int_0^\alpha \frac{f_R(k)\,dk}{k^2(\alpha^2-k^2)^{1/2}} \tag{2.37}$$

and

$$\int_0^\infty \frac{f_R(k)\,dk}{k^2} = 0. \tag{2.38}$$

Equation (2.38) can also be derived by applying the asymptotic argument as $k \to \infty$ to (2.16). Equation (2.38) indicates the existence of non-trivial zeros for the function $f_R(k)$ (for $k \neq 0$), and this has been remarked upon elsewhere (King 1975).

3. Application to the H^- ion

In this section we apply the sum rule equation (2.25) to check the consistency of the differential elastic cross section and the total cross section, which for the case of the H^- ion, can be obtained in a relatively simple closed form, within the framework of the asymptotic approximation. The derivation of the scattering amplitude essentially coincides with Adelman's (1972) derivation of the dynamic polarizability for H^-. So we just briefly sketch the derivation, leaving the reader to consult Adelman's paper for additional details on the justification of the asymptotic approximation.

The differential elastic photon scattering cross section in the dipole length formulation for the H^- ion is given by

$$\frac{d\sigma_e(\omega)}{d\Omega} = r_0^2\omega^4 c^{-4}|P(\omega)+P(-\omega)|^2(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}')^2 \tag{3.1}$$

where

$$P(\omega) = \frac{1}{3}\sum_n \frac{\langle\phi_0(r_1,r_2)|\boldsymbol{r}_1+\boldsymbol{r}_2|\phi_n(r_1,r_2)\rangle\cdot\langle\phi_n(r_1,r_2)|\boldsymbol{r}_1+\boldsymbol{r}_2|\phi_0(r_1,r_2)\rangle}{E_0-E_n+\omega}. \tag{3.2}$$

Equation (3.2) may be rewritten as

$$P(\omega) = \tfrac{1}{3}\langle\phi_0(r_1,r_2)|(\boldsymbol{r}_1+\boldsymbol{r}_2)\cdot\boldsymbol{\Psi}(r_1,r_2)\rangle \tag{3.3}$$

where

$$\boldsymbol{\Psi}(r_1,r_2) = \sum_n \frac{|\phi_n(r_1,r_2)\rangle\langle\phi_n(r_1,r_2)|\boldsymbol{r}_1+\boldsymbol{r}_2|\phi_0(r_1,r_2)\rangle}{E_0-E_n+\omega} \tag{3.4}$$

and $\boldsymbol{\Psi}(r_1,r_2)$ satisfies the differential equation

$$(E_0-H+\omega)\boldsymbol{\Psi}(r_1,r_2) = (\boldsymbol{r}_1+\boldsymbol{r}_2)\phi_0(r_1,r_2) \tag{3.5}$$

where

$$H = -\frac{1}{2}\nabla_1^2-\frac{1}{2}\nabla_2^2-\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_{12}}. \tag{3.6}$$

From equation (3.5) the following result is obtained:

$$\boldsymbol{\theta}(r_1)(E_0-E_{1s}+\omega)+\tfrac{1}{2}\nabla_1^2\boldsymbol{\theta}(r_1) = \boldsymbol{r}_1\chi(r_1) \tag{3.7}$$

where

$$\boldsymbol{\theta}(r_1) = \int \Psi_{1s}(r_2)\boldsymbol{\Psi}(r_1, r_2)\,\mathrm{d}\tau_2, \tag{3.8}$$

$$E_0 - E_{1s} + \omega = \omega - \tfrac{1}{2}\gamma^2 \tag{3.9}$$

and the asymptotic approximation has been introduced as

$$\lim_{r_1 \to \infty} \phi_0(r_1, r_2) = \chi(r_1)\Psi_{1s}(r_2) \tag{3.10}$$

where

$$\chi(r_1) = \frac{N\,\mathrm{e}^{-\gamma r_1}}{r_1} \tag{3.11}$$

and N is a normalization constant, $\gamma = (2E)^{1/2}$, where E is the binding energy. Re-expressing $\boldsymbol{\theta}(r)$ in the form

$$\boldsymbol{\theta}(r) = \hat{\boldsymbol{r}}\chi(r)g(r)r^n \tag{3.12}$$

converts (3.7) into the following differential equation:

$$g''(r) + 2\left(\frac{n}{r} - \gamma\right)g'(r) + \left(2\omega - \frac{2n\gamma}{r} + \frac{n^2 - n - 2}{r^2}\right)g(r) = 2r^{1-n} \tag{3.13}$$

which for the choice $n = -1$ can be solved by taking the Laplace transform, $G(s) = \int_0^\infty g(r)\,\mathrm{e}^{-sr}\,\mathrm{d}r$. $P(\omega)$ is then obtained directly as

$$P(\omega) = \frac{8\pi N^2}{3} G(2\gamma); \tag{3.14}$$

hence the scattering amplitude is given by

$$f(\omega) = -\tfrac{4}{3}r_0 c^{-2}\omega^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')\pi N^2\gamma^{-3}\left(\frac{[\gamma^2 + 2\gamma(\gamma^2 - 2\omega)^{1/2} - \omega]}{[\gamma + (\gamma^2 - 2\omega)^{1/2}]^2[\gamma^2 + \gamma(\gamma^2 - 2\omega)^{1/2} - \omega]}\right.$$
$$\left. + \frac{[\gamma^2 + 2\gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}{[\gamma + (\gamma^2 + 2\omega)^{1/2}]^2[\gamma^2 + \gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}\right). \tag{3.15}$$

From equation (3.15) the total cross section is obtained as

$$\sigma_{\mathrm{T}}(\omega) = 4\pi c\omega^{-1} f_{\mathrm{I}}(\omega) = \tfrac{16}{3}\pi^2 N^2 r_0 c^{-1}\omega^{-3}(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')(2\omega - \gamma^2)^{3/2}, \qquad \omega > \tfrac{1}{2}\gamma^2. \tag{3.16}$$

Also we have the results

$$f_{\mathrm{R}}(\omega) = -\tfrac{4}{3}\pi N^2\gamma^{-3}\omega^2 c^{-2} r_0(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')\left(\frac{[\gamma^2 + 2\gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}{[\gamma + (\gamma^2 + 2\omega)^{1/2}]^2[\gamma^2 + \gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}\right.$$
$$\left. + \frac{[\gamma^2 + 2\gamma(\gamma^2 - 2\omega)^{1/2} - \omega]}{[\gamma + (\gamma^2 - 2\omega)^{1/2}]^2[\gamma^2 + \gamma(\gamma^2 - 2\omega)^{1/2} - \omega]}\right), \qquad \omega \leqslant \tfrac{1}{2}\gamma^2 \tag{3.17}$$

and

$$f_{\mathrm{R}}(\omega) = -\tfrac{2}{3}\pi N^2\gamma^{-3}\omega^2 c^{-2} r_0(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')\left(\frac{2[\gamma^2 + 2\gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}{[\gamma + (\gamma^2 + 2\omega)^{1/2}]^2[\gamma^2 + \gamma(\gamma^2 + 2\omega)^{1/2} + \omega]}\right.$$
$$\left. - \frac{(\gamma^2 - \omega)(2\gamma^4 - 4\omega\gamma^2 - \omega^2)}{\omega^4}\right), \qquad \omega > \tfrac{1}{2}\gamma^2. \tag{3.18}$$

We now employ the results of equations (3.16), (3.17) and (3.18) to test

$$\frac{1}{8\pi^2c^2}\int_{\gamma^2/2}^{\infty}\sigma_{\mathrm{T}}^2(\omega)\,\mathrm{d}\omega=\int_0^{\infty}\frac{\mathrm{d}\sigma_{\mathrm{e}}(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2}. \tag{3.19}$$

The left-hand side is obtained readily,

$$\frac{1}{8\pi^2c^2}\int_{\gamma^2/2}^{\infty}\sigma_{\mathrm{T}}^2(\omega)\,\mathrm{d}\omega=\frac{256}{45}\frac{r_0^2\pi^2N^4(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2}{c^4\gamma^4}. \tag{3.20}$$

The right-hand side of (3.19) is most conveniently treated by dividing the integral into the two intervals $(0,\frac{1}{2}\gamma^2)$ and $(\frac{1}{2}\gamma^2,\infty)$. We then have

$$\int_0^{\gamma^2/2}\frac{\mathrm{d}\sigma_{\mathrm{e}}(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2}=\frac{64}{9}r_0^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2N^4c^{-4}\pi^2\gamma^{-4}\int_0^{1/2}\omega^2\,\mathrm{d}\omega$$
$$\times\left(\frac{[1+\omega+2(1+2\omega)^{1/2}]}{[1+(1+2\omega)^{1/2}]^4}+\frac{[1-\omega+2(1-2\omega)^{1/2}]}{[1+(1-2\omega)^{1/2}]^4}\right)^2. \tag{3.21}$$

Using the result

$$\int_0^{1/2}\omega^2\,\mathrm{d}\omega\left(\frac{[1+\omega+2(1+2\omega)^{1/2}]}{[1+(1+2\omega)^{1/2}]^4}+\frac{[1-\omega+2(1-2\omega)^{1/2}]}{[1+(1-2\omega)^{1/2}]^4}\right)^2$$
$$=\frac{3}{4}\left(\frac{189\sqrt{2}}{5}-\frac{254}{5}-3\ln(2^{1/2}+1)\right) \tag{3.22}$$

we obtain the following:

$$\int_0^{\gamma^2/2}\frac{\mathrm{d}\sigma_{\mathrm{e}}(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2}=\frac{16}{15}r_0^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2N^4c^{-4}\pi^2\gamma^{-4}[189\sqrt{2}-15\ln(2^{1/2}+1)-254]. \tag{3.23}$$

Also, we have that

$$\int_{\gamma^2/2}^{\infty}\frac{\mathrm{d}\sigma_{\mathrm{e}}(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2}=\pi^2N^4c^{-4}r_0^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2\gamma^{-4}$$
$$\times\left[\frac{128}{45}+\frac{4}{9}\int_{1/2}^{\infty}\omega^2\,\mathrm{d}\omega\left(\frac{[1+\omega+2(1+2\omega)^{1/2}]}{[1+\omega+(1+2\omega)^{1/2}]^2}-\frac{(1-\omega)(2-4\omega-\omega^2)}{\omega^4}\right)^2\right]. \tag{3.24}$$

Employing the result

$$\int_{1/2}^{\infty}\omega^2\,\mathrm{d}\omega\left(\frac{[1+\omega+2(1+2\omega)^{1/2}]}{[1+\omega+(1+2\omega)^{1/2}]^2}-\frac{(1-\omega)(2-4\omega-\omega^2)}{\omega^4}\right)^2$$
$$=\left(616-\frac{2268\sqrt{2}}{5}+36\ln(2^{1/2}+1)\right) \tag{3.25}$$

we have that

$$\int_{\gamma^2/2}^{\infty}\frac{\mathrm{d}\sigma_{\mathrm{e}}(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2}=\pi^2N^4c^{-4}r_0^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2\gamma^{-4}\left[\frac{128}{45}+\frac{16}{9}\left(154-\frac{567\sqrt{2}}{5}+9\ln(2^{1/2}+1)\right)\right]. \tag{3.26}$$

Therefore, combining (3.26) and (3.23) we have the result

$$\int_0^\infty \frac{\mathrm{d}\sigma_e(\omega)}{\mathrm{d}\Omega}\frac{\mathrm{d}\omega}{\omega^2} = \tfrac{256}{45}\pi^2 N^4 c^{-4} r_0^2(\boldsymbol{\epsilon}\,.\,\boldsymbol{\epsilon}')^2\gamma^{-4} \tag{3.27}$$

which is identical with (3.20). This completes the verification of the sum rule (3.19) for the H^- ion within the framework of the asymptotic approximation.

4. Discussion

Sum rules are frequently the only means by which both experimental results and theoretically derived expressions can be tested. The rather basic assumptions under which the sum rules have been derived allow them to be applied widely. The example illustrated, the H^- ion, is a relatively simple test case, because of the absence of bound excited states for this ion. The scattering amplitude for the hydrogen atom has recently been obtained in closed form (Gavrila 1967, Gavrila and Costescu 1970). The final result is, however, rather complex, and the testing of sum rules of the type such as (2.25) becomes a more difficult proposition.

The most useful application of sum rules such as (2.25) would be their utilization in the testing of experimentally derived relations for the scattering cross sections, based on *both* experimental and theoretical considerations. It is likely that if more relations of the form presented in § 2 become available, a sufficient stimulus will be provided for expressing results for the total cross section via empirical relationships, which can then be tested using the appropriate sum rules.

A number of the sum rules outlined in § 2 require the availability of the total cross section, an experimental quantity, and the differential elastic scattering cross section for the forward direction, a quantity presently accessible only through calculation. Fairly accurate configuration-interaction calculations of transition matrix elements, necessary for the calculation of the scattering amplitude, are presently restricted to principally atomic systems, with very little accurate work having been carried out on molecular systems. With this situation likely to remain for at least the next few years, it would be of value if simple theoretical expressions for the forward differential cross section could be obtained, as in the case for the H^- ion. This approach is not, however, a complete substitute for accurate calculations of the scattering amplitude at any particular desired frequency.

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References

Adelman S A 1972 *Phys. Rev.* A **5** 508–14
—— 1973 *J. Phys.* B: *Atom. Molec. Phys.* **6** 1986–91
Altarelli M, Dexter D L, Nussenzveig H M and Smith D Y 1972 *Phys. Rev.* B **6** 4502–9
Altarelli M and Smith D Y 1974 *Phys. Rev.* B **9** 1290–8
Damashek M and Gilman F J 1970 *Phys. Rev.* D **1** 1319–32

Drell S D and Hearn A C 1966 *Phys. Rev. Lett.* **16** 908–11
Ferro Fontán C, Queen N M and Violini G 1972 *Riv. Nuovo Cim.* **2** 357–497
Frye G and Warnock R L 1963 *Phys. Rev.* **130** 478–94
Gavrila M 1967 *Phys. Rev.* **163** 147–55
Gavrila M and Costescu 1970 *Phys. Rev.* A **2** 1752–8
Gell-Mann M, Goldberger M L and Thirring W E 1954 *Phys. Rev.* **95** 1612–27
Goldberger M L and Low F E 1968 *Phys. Rev.* **176** 1778–81
Granovskiĭ Ya I 1969 *Sov. Phys.–JETP* **29** 333–6
King F W 1975 *Phys. Rev.* A **11** 381–2
Kubo R and Ichimura M 1972 *J. Math. Phys.* **13** 1454–61
Liu Y C and Okubo S 1967 *Phys. Rev. Lett.* **19** 190–2
Maximon L C and O'Connell J S 1974 *Phys. Lett.* **48B** 399–402
Moses H E 1972 *Air Force Cambridge Research Laboratories, Physical Sciences Research Papers* No 490
—— 1973 *Air Force Cambridge Research Laboratories, Physical Sciences Research Papers* No 578
Villani A and Zimerman A H 1973 *Phys. Rev.* B **8** 3914–6